Wakimoto realization of Drinfeld current for the elliptic quantum algebra $U_{q,p}(\widehat{sl}_3)$

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Abstract

We study a free field realization of the elliptic quantum algebra $U_{q,p}(\widehat{sl_3})$ for arbitrary level k. We give the free field realization of elliptic analogue of Drinfeld current associated with $U_{q,p}(\widehat{sl_3})$ for arbitrary level k. In the limit $p \to 0, q \to 1$ our realization reproduces Wakimoto realization for the affine Lie algebra $\widehat{sl_3}$.

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1 Introduction

The elliptic quantum group has been proposed in papers [1, 2, 3, 4, 5]. There are two types of elliptic quantum groups, the vertex type $\mathcal{A}_{q,p}(\widehat{sl_N})$ and the face type $\mathcal{B}_{q,\lambda}(g)$, where g is a Kac-Moody algebra associated with a symmetrizable Cartan matrix. The elliptic quantum groups have the structure of quasi-triangular quasi-Hopf algebras introduced by V.Drinfeld [6]. H.Konno [7] introduced the elliptic quantum algebra $U_{q,p}(\widehat{sl_2})$ as an algebra of the screening currents of the extended deformed Virasoro algebra in terms of the fusion SOS model [8]. M.Jimbo, H.Konno, S.Odake, J.Shiraishi [9] continued to

study the elliptic quantum algebra $U_{q,p}(\widehat{sl_2})$. They constructed the elliptic alnalogue of Drinfeld currents and identified $U_{q,p}(\widehat{sl_2})$ with the tensor product of $\mathcal{B}_{q,\lambda}(\widehat{sl_2})$ and a Heisenberg algebra $\mathcal{H}$. The elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{sl_2})$ is a quasi-Hopf algebra while the elliptic algebra $U_{q,p}(\widehat{sl_2})$ is not. The intertwining relation of the vertex operator of $\mathcal{B}_{q,\lambda}(\widehat{sl_2})$ is based on the quasi-Hopf structure of $\mathcal{B}_{q,\lambda}(\widehat{sl_2})$. By the above isomorphism $U_{q,p}(\widehat{sl_2}) \simeq \mathcal{B}_{q,\lambda}(\widehat{sl_2}) \otimes \mathcal{H}$, we can understand "intertwining relation" of the vertex operator for the elliptic algebra $U_{q,p}(\widehat{sl_2})$. Along the above scheme the elliptic analogue of Drinfeld current of $U_{q,p}(\widehat{sl_2})$ is extended to those of $U_{q,p}(g)$ for non-twisted affine Lie algebra g [9, 10]. In this paper we are interested in higher-rank generalization of level k free field realization of the elliptic quantum algebra. For the elliptic algebra $U_{q,p}(\widehat{sl_2})$, there exist two kind of free field realizations for arbitrary level k, the one is parafermion realization [7, 9], the other is Wakimoto realization [16]. In this paper we are interested in the higher-rank generalization of Wakimoto realization of $U_{q,p}(\widehat{sl_2})$. We construct level k free field realization of Drinfeld current associated with the elliptic algebra $U_{q,p}(\widehat{sl_3})$. This gives the first example of arbitrary level free field realization of the higher-rank elliptic algebra. This free field realization can be applied for construction of the integrals of motion for the elliptic algebra $U_{q,p}(\widehat{sl_3})$. For this purpose, see references [17, 18, 19].

The organization of this paper is as follows. In section 2 we set the notation and introduce bosons. In section 3 we review the level k free field realization of the quantum group $U_q(\widehat{sl_3})$ [15]. In section 4 we give the level k free field realization of the elliptic quantum algebra $U_{q,p}(\widehat{sl_3})$. In appendix we summarize the normal ordering of the basic operators.

2 Boson

The purpose of this section is to set up the basic notation and to introduce the boson. In this paper we fix three parameters $q, k, r \in \mathbb{C}$. Let us set $r^* = r - k$. We assume $k \neq 0, -3$ and $\mathrm{Re}(r) > 0$, $\mathrm{Re}(r^*) > 0$. We assume q is a generic with $|q| < 1, q \neq 0$. Let us set a pair of parameters p and p^* by

$$p = q^{2r}, \quad p^* = q^{2r^*}.$$

We use the standard symbol of q-integer $[n]$ by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Let us set the elliptic theta function $\Theta_p(z)$ by

$$\Theta_p(z) = (z;p)_\infty (p/z;p)_\infty (p;p)_\infty,$$
$$(z;p)_\infty = \prod_{n=0}^{\infty} (1 - p^n z).$$

It is convenient to work with the additive notation. We use the parametrization

$$\begin{aligned} q &= e^{-\pi\sqrt{-1}/r\tau}, \\ p &= e^{-2\pi\sqrt{-1}/\tau}, \quad p^* = e^{-2\pi\sqrt{-1}/\tau^*}, \quad (r\tau = r^*\tau^*), \\ z &= q^{2u}. \end{aligned}$$

Let us set Jacobi elliptic theta function $[u]_r, [u]_{r^*}$ by

$$[u]_r = q^{\frac{u^2}{r} - u} \frac{\Theta_p(z)}{(p;p)_\infty^3}, \quad [u]_{r^*} = q^{\frac{u^2}{r^*} - u} \frac{\Theta_{p^*}(z)}{(p^*;p^*)_\infty^3}.$$

The function $[u]_r$ has a zero at $u = 0$, enjoys the quasi-periodicity property

$$[u + r]_r = -[u]_r, \qquad [u + r\tau]_r = -e^{-\pi\sqrt{-1}\tau - \frac{2\pi\sqrt{-1}u}{r}} [u]_r.$$

Let us set the delta-function $\delta(z)$ as formal power series.

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n.$$

Following [15] we introduce free bosons $a_n^1, a_n^2, b_n^1, b_n^2, b_n^3, c_n^1, c_n^2, c_n^3, (n \in \mathbb{Z}_{\neq 0})$.

$$[a_n^i, a_m^j] = \frac{[(k+3)n][A_{i,j}n]}{n} \delta_{n+m,0}, \quad [p_a^i, q_a^j] = (k+3)A_{i,j}, \quad (i,j = 1,2), \tag{2.1}$$

$$[b_n^i, b_m^j] = -\frac{[n]^2}{n} \delta_{i,j} \delta_{n+m,0}, \quad [p_b^i, q_b^j] = -\delta_{i,j}, \quad (i,j = 1,2,3), \tag{2.2}$$

$$[c_n^i, c_m^j] = \frac{[n]^2}{n} \delta_{i,j} \delta_{n+m,0}, \quad [p_c^i, q_c^j] = \delta_{i,j}, \quad (i,j = 1,2,3). \tag{2.3}$$

Here we have used Cartan matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

For parameters $a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$, we set the vacuum vector $|a,b,c\rangle$ of the Fock space $\mathcal{F}_{a_1 a_2 b_1 b_2 b_3 c_1 c_2 c_3}$ as following.

$$a_n^i |a,b,c\rangle = b_n^j |a,b,c\rangle = c_n^j |a,b,c\rangle = 0, \quad (i = 1,2; j = 1,2,3), \tag{2.4}$$

$$p_a^i|a,b,c\rangle = a_i|a,b,c\rangle,\ p_b^j|a,b,c\rangle = b_j|a,b,c\rangle,\ p_c^j|a,b,c\rangle = c_j|a,b,c\rangle,$$
$$(i=1,2; j=1,2,3; n>0). \tag{2.5}$$

The Fock space $\mathcal{F}_{a_1a_2b_1b_2b_3c_1c_2c_3}$ is generated by bosons $a_{-n}^1, a_{-n}^2, b_{-n}^1, b_{-n}^2, b_{-n}^3, c_{-n}^1, c_{-n}^2, c_{-n}^3$ for $n \in \mathbb{N}_{\neq 0}$. The dual Fock space $\mathcal{F}^*_{a_1a_2b_1b_2b_3c_1c_2c_3}$ is defined as the same manner. In this paper we construct the elliptic analogue of Drinfeld current for $U_{q,p}(\widehat{sl_3})$ by these bosons a_n^i, b_n^j, c_n^j acting on the Fock space.

3 Free Field Realization of $U_q(\widehat{sl_3})$

The purpose of this section is to give the free field realization of the quantum affine algebra $U_q(\widehat{sl_3})$. We give a review of Wakimoto realization of $U_q(\widehat{sl_3})$ [15]. Let us set the bosonic operators $a_\pm^i(z), b_\pm^i(z), \gamma^i(z), \beta_s^i(z)$ by

$$\begin{aligned}
a_\pm^i(z) &= \pm(q-q^{-1})\sum_{n>0} a_{\pm n}^i z^{\mp n} \pm p_a^i \log q, \quad (i=1,2), && (3.1)\\
b_\pm^i(z) &= \pm(q-q^{-1})\sum_{n>0} b_{\pm n}^i z^{\mp n} \pm p_b^i \log q, \quad (i=1,2,3), && (3.2)\\
b^i(z) &= -\sum_{n\neq 0} \frac{b_n^i}{[n]} z^{-n} + q_b^i + p_b^i \log z, \quad (i=1,2,3), && (3.3)\\
c^i(z) &= -\sum_{n\neq 0} \frac{c_n^i}{[n]} z^{-n} + q_c^i + p_c^i \log z, \quad (i=1,2,3), && (3.4)\\
\gamma^i(z) &= -\sum_{n\neq 0} \frac{(b+c)_n^i}{[n]} z^{-n} + (q_b^i+q_c^i) + (p_b^i+p_c^i)\log(-z), \quad (i=1,2,3), && (3.5)\\
\beta_1^i(z) &= b_+^i(z) - (b^i+c^i)(qz),\ \beta_2^i(z) = b_-^i(z) - (b^i+c^i)(q^{-1}z), \quad (i=1,2,3), && (3.6)\\
\beta_1^i(z) &= b_+^i(z) + (b^i+c^i)(qz),\ \beta_2^i(z) = b_-^i(z) + (b^i+c^i)(q^{-1}z), \quad (i=1,2,3). && (3.7)
\end{aligned}$$

We give a free field realiztaion of Drinfeld current for $U_q(\widehat{sl_3})$.

Definition 3.1 *We define the bosonic operators $e_1^+(z), e_2^+(z), e_1^-(z), e_2^-(z)$ by*

$$\begin{aligned}
e_1^+(z) &= \frac{-1}{(q-q^{-1})z}(e_1^{+,1}(z) - e_1^{+,2}(z)), && (3.8)\\
e_2^+(z) &= \frac{-1}{(q-q^{-1})z}(e_2^{+,1}(z) - e_2^{+,2}(z) + e_2^{+,3}(z) - e_2^{+,4}(z)), && (3.9)\\
e_1^-(z) &= \frac{-1}{(q-q^{-1})z}(e_1^{-,1}(z) - e_1^{-,2}(z) - e_1^{-,3}(z) + e_1^{-,4}(z)), && (3.10)\\
e_2^-(z) &= \frac{-1}{(q-q^{-1})z}(e_2^{-,1}(z) - e_2^{-,2}(z) + e_2^{-,3}(z) - e_2^{-,4}(z)). && (3.11)
\end{aligned}$$

$$\psi_1^{\pm}(z) = :\exp\left(b_\pm^1(q^{\pm k}z)+b_\pm^1(q^{\pm(k+2)}z)+b_\pm^2(q^{\pm(k+3)}z)-b_\pm^3(q^{\pm(k+2)}z)+a_\pm^1(q^{\pm\frac{k+3}{2}}z)\right):, \tag{3.12}$$

$$\psi_2^{\pm}(z) = :\exp\left(-b_\pm^1(q^{\pm(k+1)}z)+b_\pm^2(q^{\pm k}z)+b_\pm^3(q^{\pm(k+1)}z)+b_\pm^3(q^{\pm(k+3)}z)+a_\pm^2(q^{\pm\frac{k+3}{2}}z)\right):, \tag{3.13}$$

Here we have set

$$e_1^{+,1}(z) = :\exp\left(\beta_1^1(z)\right):, \tag{3.14}$$
$$e_1^{+,2}(z) = :\exp\left(\beta_2^1(z)\right):, \tag{3.15}$$
$$e_2^{+,1}(z) = :\exp\left(\gamma^1(z)+\beta_1^2(z)\right):, \tag{3.16}$$
$$e_2^{+,2}(z) = :\exp\left(\gamma^1(z)+\beta_2^2(z)\right):, \tag{3.17}$$
$$e_2^{+,3}(z) = :\exp\left(\beta_1^3(qz)+b_+^2(z)-b_+^1(qz)\right):, \tag{3.18}$$
$$e_2^{+,4}(z) = :\exp\left(\beta_2^3(qz)+b_+^2(z)-b_+^1(qz)\right):, \tag{3.19}$$

$$e_1^{-,1}(z) = :\exp\left(\beta_4^1(q^{-k-2}z)+b_-^2(q^{-k-3}z)-b_-^3(q^{-k-2}z)+a_-^1(q^{-\frac{k+3}{2}}z)\right):, \tag{3.20}$$
$$e_1^{-,2}(z) = :\exp\left(\beta_3^1(q^{k+2}z)+b_+^2(q^{k+3}z)-b_+^3(q^{k+2}z)+a_+^1(q^{\frac{k+3}{2}}z)\right):, \tag{3.21}$$
$$e_1^{-,3}(z) = :\exp\left(\gamma^2(q^{k+2}z)+\beta_1^3(q^{k+2}z)+b_+^2(q^{k+3}z)-b_+^3(q^{k+2}z)+a_+^1(q^{\frac{k+3}{2}}z\right):, \tag{3.22}$$
$$e_1^{-,4}(z) = :\exp\left(\gamma^2(q^{k+2}z)+\beta_2^3(q^{k+2}z)+b_+^2(q^{k+3}z)-b_+^3(q^{k+2}z)+a_+^1(q^{\frac{k+3}{2}}z\right):, \tag{3.23}$$
$$e_2^{-,1}(z) = :\exp\left(\gamma^2(q^{-k-1}z)-\beta_3^1(q^{-k-1}z)+2b_-^3(q^{-k-1}z)+a_-^2(q^{-\frac{k+3}{2}}z)\right):, \tag{3.24}$$
$$e_2^{-,2}(z) = :\exp\left(\gamma^2(q^{-k-1}z)-\beta_4^1(q^{-k-1}z)+2b_-^3(q^{-k-1}z)+a_-^2(q^{-\frac{k+3}{2}}z)\right):, \tag{3.25}$$
$$e_2^{-,3}(z) = :\exp\left(\beta_4^3(q^{-k-3}z)+a_-^2(q^{-\frac{k+3}{2}}z)\right):, \tag{3.26}$$
$$e_2^{-,4}(z) = :\exp\left(\beta_3^3(q^{k+3}z)+a_+^2(q^{\frac{k+3}{2}}z)\right):. \tag{3.27}$$

Here the symbol $:\mathcal{O}:$ represents the normal ordering of $\mathcal{O}$. For example we have

$$:b_k b_l:=\begin{cases} b_k^i b_l^i, & k<0 \\ b_l^i b_k^i, & k>0. \end{cases} \qquad :p_b^i q_b^i:=:q_b^i p_b^i:=q_b^i p_b^i.$$

Theorem 3.1 *[15] The bosonic operators $e_i^{\pm}(z)$, $\psi_i^{\pm}(z)$, $(i=1,2)$ satisfy the following commutation relations.*

$$(z_1-q^{A_{i,j}}z_2)e_i^+(z_1)e_j^+(z_2) = (q^{A_{i,j}}z_1-z_2)e_j^+(z_2)e_i^+(z_1), \tag{3.28}$$

$$(z_1-q^{-A_{i,j}}z_2)e_i^-(z_1)e_j^-(z_2) = (q^{-A_{i,j}}z_1-z_2)e_j^-(z_2)e_i^-(z_1), \tag{3.29}$$

$$[\psi_i^\pm(z_1),\psi_j^\pm(z_2)] = 0, \tag{3.30}$$

$$\begin{aligned}&(z_1-q^{A_{i,j}-k}z_2)(z_1-q^{-A_{i,j}+k}z_2)\psi_i^\pm(z_1)\psi_j^\mp(z_2)\\ =\ &(z_1-q^{A_{i,j}+k}z_2)(z_1-q^{-A_{i,j}-k}z_2)\psi_j^\mp(z_2)\psi_i^\pm(z_1),\end{aligned} \tag{3.31}$$

$$(z_1-q^{\pm(A_{i,j}-\frac{k}{2})}z_2)\psi_i^+(z_1)e_j^\pm(z_2) = (q^{\pm A_{i,j}}z_1-q^{\mp\frac{k}{2}}z_2)e_j^\pm(z_2)\psi_i^+(z_1), \tag{3.32}$$

$$(z_1-q^{\pm(A_{i,j}-\frac{k}{2})}z_2)e_i^\pm(z_1)\psi_j^-(z_2) = (q^{\pm A_{i,j}}z_1-q^{\mp\frac{k}{2}}z_2)\psi_j^-(z_2)e_i^\pm(z_1), \tag{3.33}$$

$$\begin{aligned}&\left\{e_i^\pm(z_1)e_i^\pm(z_2)e_j^\pm(z_3)-(q+q^{-1})e_i^\pm(z_1)e_j^\pm(z_3)e_j^\pm(z_2)+e_i^\pm(z_3)e_i^\pm(z_1)e_j^\pm(z_2)\right\}\\ &+\{z_1\leftrightarrow z_2\}=0,\quad \text{for}\quad (i\neq j),\end{aligned} \tag{3.34}$$

$$[e_i^+(z_1),e_j^-(z_2)]=\frac{\delta_{i,j}}{(q-q^{-1})z_1z_2}\left(\delta\left(q^{-k}\frac{z_1}{z_2}\right)\psi_i^+(q^{-\frac{k}{2}}z_1)-\delta\left(q^{k}\frac{z_1}{z_2}\right)\psi_i^-(q^{-\frac{k}{2}}z_2)\right). \tag{3.35}$$

Hence $e_i^\pm(z),\psi_i^\pm(z)$ give level k free field realization of $U_q(\widehat{sl_3})$.

4 Free Field Realization of $U_{q,p}(\widehat{sl_3})$

The purpose of this section is to give a free field realization of the elliptic analogue of Drinfeld current for $U_{q,p}(\widehat{sl_3})$ with arbitrary level $k\neq 0,-3$. Let us set the bosonic operators $\mathcal{B}_\pm^{*i}(z),\mathcal{B}_\pm^i(z),(i=1,2,3)$, $\mathcal{A}^{*i}(z),\mathcal{A}^i(z),(i=1,2)$ by

$$\mathcal{B}_\pm^{*i}(z) = \exp\left(\pm\sum_{n>0}\frac{b_{-n}^i}{[r^*n]}z^n\right),\quad (i=1,2,3), \tag{4.1}$$

$$\mathcal{B}_\pm^{i}(z) = \exp\left(\pm\sum_{n>0}\frac{b_{n}^i}{[rn]}z^{-n}\right),\quad (i=1,2,3), \tag{4.2}$$

$$\mathcal{A}^{i*}(z) = \exp\left(\sum_{n>0}\frac{a_{-n}^i}{[r^*n]}z^n\right),\quad (i=1,2), \tag{4.3}$$

$$\mathcal{A}^{i}(z) = \exp\left(-\sum_{n>0}\frac{a_{n}^i}{[rn]}z^{-n}\right),\quad (i=1,2). \tag{4.4}$$

Definition 4.1 *Let us set the bosonic operators* $e_i(z), f_i(z), \Psi_i^{\pm}(z), (i=1,2)$ *by*

$$e_i(z) = U^{*i}(z)e_i^{+}(z), \quad (i=1,2), \tag{4.5}$$

$$f_i(z) = e_i^{-}(z)U^i(z), \quad (i=1,2), \tag{4.6}$$

$$\Psi_i^{+}(z) = U^{*i}(q^{\frac{k}{2}}z)\psi_i^{+}(z)U^i(q^{-\frac{k}{2}}z), \quad (i=1,2), \tag{4.7}$$

$$\Psi_i^{+}(z) = U^{*i}(q^{-\frac{k}{2}}z)\psi_i^{-}(z)U^i(q^{\frac{k}{2}}z), \quad (i=1,2). \tag{4.8}$$

Here we have set

$$U^{*1}(z) = \mathcal{B}_{+}^{*1}(q^{r^*}z)\mathcal{B}_{+}^{*1}(q^{r^*-2}z)\mathcal{B}_{+}^{*2}(q^{r^*-3}z)\mathcal{B}_{-}^{*3}(q^{r^*-2}z)\mathcal{A}^{*1}(q^{r^*+\frac{k-3}{2}}z), \tag{4.9}$$

$$U^{*2}(z) = \mathcal{B}_{+}^{*3}(q^{r^*-3}z)\mathcal{B}_{+}^{*3}(q^{r^*-1}z)\mathcal{B}_{+}^{*2}(q^{r^*}z)\mathcal{B}_{-}^{*1}(q^{r^*-1}z)\mathcal{A}^{*2}(q^{r^*+\frac{k-3}{2}}z), \tag{4.10}$$

$$U^{1}(z) = \mathcal{B}_{-}^{1}(q^{-r^*}z)\mathcal{B}_{-}^{1}(q^{-r^*+2}z)\mathcal{B}_{-}^{2}(q^{-r^*+3}z)\mathcal{B}_{+}^{3}(q^{-r^*+2}z)\mathcal{A}^{1}(q^{-r^*-\frac{k-3}{2}}z), \tag{4.11}$$

$$U^{2}(z) = \mathcal{B}_{-}^{3}(q^{-r^*+1}z)\mathcal{B}_{-}^{3}(q^{-r^*+1}z)\mathcal{B}_{-}^{2}(q^{-r^*}z)\mathcal{B}_{+}^{1}(q^{-r^*+1}z)\mathcal{A}^{2}(q^{-r^*-\frac{k-3}{2}}z). \tag{4.12}$$

The above free field realization of the twistors $U^{*i}(z), U^i(z)$, $(i=1,2)$ is the main result of this paper.

Proposition 4.1 *The bosonic operators* $e_i(z), f_i(z), \Psi_i^{\pm}(z)$, $(i=1,2)$ *satisfy the following commutation relations.*

$$e_i(z_1)e_j(z_2) = q^{-A_{i,j}}\frac{\Theta_{p^*}(q^{A_{i,j}}z_1/z_2)}{\Theta_{p^*}(q^{-A_{i,j}}z_1/z_2)}e_j(z_2)e_i(z_1), \tag{4.13}$$

$$f_i(z_1)f_j(z_2) = q^{A_{i,j}}\frac{\Theta_{p}(q^{-A_{i,j}}z_1/z_2)}{\Theta_{p}(q^{A_{i,j}}z_1/z_2)}f_j(z_2)f_i(z_1), \tag{4.14}$$

$$\Psi_i^{\pm}(z_1)\Psi_j^{\pm}(z_2) = \frac{\Theta_p(q^{-A_{i,j}}z_1/z_2)\Theta_{p^*}(q^{A_{i,j}}z_1/z_2)}{\Theta_p(q^{A_{i,j}}z_1/z_2)\Theta_{p^*}(q^{-A_{i,j}}z_1/z_2)}\Psi_j^{\pm}(z_2)\Psi_i^{\pm}(z_1), \tag{4.15}$$

$$\Psi_i^{\pm}(z_1)\Psi_j^{\mp}(z_2) = \frac{\Theta_p(pq^{-A_{i,j}-k}z_1/z_2)\Theta_{p^*}(p^*q^{A_{i,j}+k}z_1/z_2)}{\Theta_p(pq^{A_{i,j}-k}z_1/z_2)\Theta_{p^*}(p^*q^{-A_{i,j}+k}z_1/z_2)}\Psi_j^{\mp}(z_2)\Psi_i^{\pm}(z_1), \tag{4.16}$$

$$\Psi_i^{\pm}(z_1)e_j(z_2) = \frac{\Theta_{p^*}(q^{A_{i,j}\pm\frac{k}{2}}z_1/z_2)}{\Theta_{p^*}(q^{-A_{i,j}\pm\frac{k}{2}}z_1/z_2)}e_j(z_2)\Psi_i^{\pm}(z_1), \tag{4.17}$$

$$\Psi_i^{\pm}(z_1)f_j(z_2) = \frac{\Theta_{p^*}(q^{-A_{i,j}\mp\frac{k}{2}}z_1/z_2)}{\Theta_{p^*}(q^{A_{i,j}\mp\frac{k}{2}}z_1/z_2)}e_j(z_2)\Psi_i^{\pm}(z_1), \tag{4.18}$$

$$[e_i(z_1), f_j(z_2)] = \frac{\delta_{i,j}}{(q-q^{-1})z_1z_2}\left(\delta\left(q^{-k}\frac{z_1}{z_2}\right)\Psi_i^{+}(q^{-k/2}z_1) - \delta\left(q^{k}\frac{z_1}{z_2}\right)\Psi_i^{-}(q^{-k/2}z_2)\right),$$
$$(i \neq j). \tag{4.19}$$

We introduce the Heisenberg algebra $\mathcal{H}$ generated by the following P_i, Q_i, $(i=1,2)$.

$$[P_i, Q_j] = \frac{A_{i,j}}{2}, \quad (i,j=1,2). \tag{4.20}$$

Definition 4.2 *Let us define the bosonic operators $E_i(z), F_i(z), H_i^{\pm}(z) \in U_q(\widehat{sl_3}) \otimes \mathcal{H}$, $(i=1,2)$ by*

$$\begin{aligned} E_1(z) &= e_1(z) e^{2Q_1} z^{-\frac{P_1-1}{r^*}}, \quad E_2(z) = e_2(z) e^{2Q_2} z^{-\frac{P_2-1}{r^*}}, && (4.21)\\ F_1(z) &= f_1(z) z^{\frac{2p_b^1+p_b^2-p_b^3+p_a^1}{r}} z^{\frac{P_1-1}{r}}, \quad F_2(z) = f_2(z) z^{\frac{2p_b^3+p_b^2-p_b^1+p_a^2}{r}} z^{\frac{P_2-1}{r}}, && (4.22)\\ H_1^{\pm}(z) &= \Psi_1^{\pm}(z) e^{2Q_1} (q^{\mp\frac{k}{2}} z)^{\frac{2p_b^1+p_b^2-p_b^3+p_a^1}{r}} (q^{\pm(r-\frac{k}{2})} z)^{\frac{P_1-1}{r} - \frac{P_1-1}{r^*}}, && (4.23)\\ H_2^{\pm}(z) &= \Psi_2^{\pm}(z) e^{2Q_2} (q^{\mp\frac{k}{2}} z)^{\frac{2p_b^3+p_b^2-p_b^1+p_a^2}{r}} (q^{\pm(r-\frac{k}{2})} z)^{\frac{P_2-1}{r} - \frac{P_2-1}{r^*}}. && (4.24) \end{aligned}$$

Theorem 4.2 *The bosonic operators $E_i(z), F_i(z), H_i^{\pm}(z)$, $(i=1,2)$ satisfy the following commutation relations.*

$$E_i(z_1)E_j(z_2) = \frac{\left[u_1-u_2+\frac{A_{i,j}}{2}\right]_{r^*}}{\left[u_1-u_2-\frac{A_{i,j}}{2}\right]_{r^*}} E_j(z_2)E_i(z_1), \tag{4.25}$$

$$F_i(z_1)F_j(z_2) = \frac{\left[u_1-u_2-\frac{A_{i,j}}{2}\right]_{r}}{\left[u_1-u_2+\frac{A_{i,j}}{2}\right]_{r}} F_j(z_2)F_i(z_1), \tag{4.26}$$

$$H_i^{\pm}(z_1)H_j^{\pm}(z_2) = \frac{\left[u_1-u_2-\frac{A_{i,j}}{2}\right]_{r}\left[u_1-u_2+\frac{A_{i,j}}{2}\right]_{r^*}}{\left[u_1-u_2+\frac{A_{i,j}}{2}\right]_{r}\left[u_1-u_2-\frac{A_{i,j}}{2}\right]_{r^*}} H_j^{\pm}(z_2)H_i^{\pm}(z_1), \tag{4.27}$$

$$H_i^{+}(z_1)H_j^{-}(z_2) = \frac{\left[u_1-u_2-\frac{A_{i,j}}{2}-\frac{k}{2}\right]_{r}\left[u_1-u_2+\frac{A_{i,j}}{2}+\frac{k}{2}\right]_{r^*}}{\left[u_1-u_2+\frac{A_{i,j}}{2}-\frac{k}{2}\right]_{r}\left[u_1-u_2-\frac{A_{i,j}}{2}+\frac{k}{2}\right]_{r^*}} H_j^{-}(z_2)H_i^{+}(z_1), \tag{4.28}$$

$$H_i^{\pm}(z_1)E_j(z_2) = \frac{\left[u_1-u_2\pm\frac{k}{4}+\frac{A_{i,j}}{2}\right]_{r^*}}{\left[u_1-u_2\pm\frac{k}{4}-\frac{A_{i,j}}{2}\right]_{r^*}} E_j(z_2)H_i^{\pm}(z_1), \tag{4.29}$$

$$H_i^{\pm}(z_1)F_j(z_2) = \frac{\left[u_1-u_2\mp\frac{k}{4}-\frac{A_{i,j}}{2}\right]_{r}}{\left[u_1-u_2\mp\frac{k}{4}+\frac{A_{i,j}}{2}\right]_{r}} F_j(z_2)H_i^{\pm}(z_1), \tag{4.30}$$

$$[E_i(z_1), F_j(z_2)] = \frac{\delta_{i,j}}{(q-q^{-1})z_1z_2}\left(\delta\left(q^{-k}\frac{z_1}{z_2}\right)H_i^+(q^{-\frac{k}{2}}z_1) - \delta\left(q^{k}\frac{z_1}{z_2}\right)H_i^-(q^{-\frac{k}{2}}z_2)\right) \quad (4.31)$$

Now we have costructed level k free field realization of Drinfeld current $E_i(z), F_i(z), H_i^{\pm}(z)$ for the elliptic algebra $U_{q,p}(\widehat{sl_3})$. This gives the first example of arbitrary-level free field realization of higher-rank elliptic algebra.

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Appendix

In appendix we summarize the normal ordering of the basic operators.

$$\begin{aligned}
:e^{\gamma^i(z_1)}:\mathcal{B}_+^{*i}(z_2) &= :e^{\gamma^i(z_1)}\mathcal{B}_+^{*i}(z_2): \frac{(q^{r^*+1}z_2/z_1;p^*)_\infty}{(q^{r^*-1}z_2/z_1;p^*)_\infty},\\
:e^{\beta_1^i(z_1)}:\mathcal{B}_+^{*i}(z_2) &= :e^{\beta_1^i(z_1)}\mathcal{B}_+^{*i}(z_2): \frac{(q^{r^*}z_2/z_1;p^*)_\infty}{(q^{r^*+2}z_2/z_1;p^*)_\infty},\\
:e^{\beta_2^i(z_1)}:\mathcal{B}_+^{*i}(z_2) &= :e^{\beta_2^i(z_1)}\mathcal{B}_+^{*i}(z_2): \frac{(q^{r^*}z_2/z_1;p^*)_\infty}{(q^{r^*+2}z_2/z_1;p^*)_\infty},\\
:e^{\beta_3^i(z_1)}:\mathcal{B}_+^{*i}(z_2) &= :e^{\beta_3^i(z_1)}\mathcal{B}_+^{*i}(z_2): \frac{(q^{r^*}z_2/z_1;p^*)_\infty}{(q^{r^*-2}z_2/z_1;p^*)_\infty},\\
:e^{\beta_4^i(z_1)}:\mathcal{B}_+^{*i}(z_2) &= :e^{\beta_4^i(z_1)}\mathcal{B}_+^{*i}(z_2): \frac{(q^{r^*}z_2/z_1;p^*)_\infty}{(q^{r^*+2}z_2/z_1;p^*)_\infty},\\
\mathcal{B}_-^i(z_1):e^{\gamma^i(z_2)}: &= :\mathcal{B}_-^i(z_1)e^{\gamma^i(z_2)}: \frac{(q^{r+1}z_2/z_1;p)_\infty}{(q^{r-1}z_2/z_1;p)_\infty},\\
\mathcal{B}_-^i(z_1):e^{\beta_1^i(z_2)}: &= :\mathcal{B}_-^i(z_1)e^{\beta_1^i(z_2)}: \frac{(q^{r}z_2/z_1;p)_\infty}{(q^{r+2}z_2/z_1;p)_\infty},\\
\mathcal{B}_-^i(z_1):e^{\beta_2^i(z_2)}: &= :\mathcal{B}_-^i(z_1)e^{\beta_2^i(z_2)}: \frac{(q^{r}z_2/z_1;p)_\infty}{(q^{r+2}z_2/z_1;p)_\infty},\\
\mathcal{B}_-^i(z_1):e^{\beta_3^i(z_1)}: &= :\mathcal{B}_-^i(z_1)e^{\beta_3^i(z_1)}: \frac{(q^{r}z_2/z_1;p)_\infty}{(q^{r-2}z_2/z_1;p)_\infty},
\end{aligned}$$

$$\mathcal{B}_-^i(z_1):e^{\beta_4^i(z_2)}: \;=\; :\mathcal{B}_-^i(z_1)e^{\beta_4^i(z_2)}:\frac{(q^r z_2/z_1;p)_\infty}{(q^{r-2}z_2/z_1;p)_\infty},$$

$$\begin{aligned}
e^{b_+^i(z_1)}\mathcal{B}_+^{*i}(z_2) &= :e^{b_+^i(z_1)}\mathcal{B}_+^{*i}(z_2):\frac{(q^{r^*}z_2/z_1;p^*)_\infty^2}{(q^{r^*+2}z_2/z_1;p^*)_\infty(q^{r^*-2}z_2/z_1;p^*)_\infty},\\
\mathcal{B}_-^i(z_1)e^{b_-^i(z_2)} &= :\mathcal{B}_-^i(z_1)e^{b_-^i(z_2)}:\frac{(q^r z_2/z_1;p)_\infty^2}{(q^{r+2}z_2/z_1;p)_\infty(q^{r-2}z_2/z_1;p)_\infty},\\
e^{a_+^i(z_1)}\mathcal{A}^{*i}(z_2) &= :e^{a_+^i(z_1)}\mathcal{A}^{*i}(z_2):\frac{(q^{r^*+k+5}z_2/z_1;p^*)_\infty(q^{r^*-k-5}z_2/z_1;p^*)_\infty}{(q^{r^*+k+1}z_2/z_1;p^*)_\infty(q^{r^*-k-1}z_2/z_1;p^*)_\infty},\\
e^{a_+^1(z_1)}\mathcal{A}^{*2}(z_2) &= :e^{a_+^1(z_1)}\mathcal{A}^{*2}(z_2):\frac{(q^{r^*+k+2}z_2/z_1;p^*)_\infty(q^{r^*-k-2}z_2/z_1;p^*)_\infty}{(q^{r^*+k+4}z_2/z_1;p^*)_\infty(q^{r^*-k-4}z_2/z_1;p^*)_\infty},\\
e^{a_+^2(z_1)}\mathcal{A}^{*1}(z_2) &= :e^{a_+^2(z_1)}\mathcal{A}^{*1}(z_2):\frac{(q^{r^*+k+2}z_2/z_1;p^*)_\infty(q^{r^*-k-2}z_2/z_1;p^*)_\infty}{(q^{r^*+k+4}z_2/z_1;p^*)_\infty(q^{r^*-k-4}z_2/z_1;p^*)_\infty},\\
\mathcal{A}^i(z_1)e^{a_-^i(z_2)} &= :\mathcal{A}^i(z_1)e^{a_-^i(z_2)}:\frac{(q^{r+k+5}z_2/z_1;p)_\infty(q^{r-k-5}z_2/z_1;p)_\infty}{(q^{r+k+1}z_2/z_1;p)_\infty(q^{r-k-1}z_2/z_1;p)_\infty},\\
\mathcal{A}^1(z_1)e^{a_-^2(z_2)} &= :\mathcal{A}^1(z_1)e^{a_-^2(z_2)}:\frac{(q^{r+k+2}z_2/z_1;p)_\infty(q^{r-k-2}z_2/z_1;p)_\infty}{(q^{r+k+4}z_2/z_1;p)_\infty(q^{r-k-4}z_2/z_1;p)_\infty},\\
\mathcal{A}^2(z_1)e^{a_-^1(z_2)} &= :\mathcal{A}^2(z_1)e^{a_-^1(z_2)}:\frac{(q^{r+k+2}z_2/z_1;p)_\infty(q^{r-k-2}z_2/z_1;p)_\infty}{(q^{r+k+4}z_2/z_1;p)_\infty(q^{r-k-4}z_2/z_1;p)_\infty},
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_-^i(z_1)\mathcal{B}_+^{*i}(z_2) &= :\mathcal{B}_-^i(z_1)\mathcal{B}_+^{*i}(z_2):\frac{(q^k z_2/z_1;q^{2k},p^*)_\infty^2}{(q^{k+2}z_2/z_1;q^{2k},p^*)_\infty(q^{k-2}z_2/z_1;q^{2k},p^*)_\infty}\\
&\times\ \frac{(q^{k+2}z_2/z_1;q^{2k},p)_\infty(q^{k-2}z_2/z_1;q^{2k},p)_\infty}{(q^k z_2/z_1;q^{2k},p)_\infty^2},\\
\mathcal{A}^i(z_1)\mathcal{A}^{*i}(z_2) &= :\mathcal{A}^i(z_1)\mathcal{A}^{*i}(z_2):\frac{(q^{2k+5}z_2/z_1;q^{2k},p^*)_\infty(q^{-5}z_2/z_1;q^{2k},p^*)_\infty}{(q^{2k+1}z_2/z_1;q^{2k},p^*)_\infty(q^{-1}z_2/z_1;q^{2k},p^*)_\infty}\\
&\times\ \frac{(q^{2k+1}z_2/z_1;q^{2k},p)_\infty(q^{-1}z_2/z_1;q^{2k},p)_\infty}{(q^{2k+5}z_2/z_1;q^{2k},p)_\infty(q^{-5}z_2/z_1;q^{2k},p)_\infty},\\
\mathcal{A}^1(z_1)\mathcal{A}^{*2}(z_2) &= :\mathcal{A}^1(z_1)\mathcal{A}^{*2}(z_2):\frac{(q^{2k+2}z_2/z_1;q^{2k},p^*)_\infty(q^{-2}z_2/z_1;q^{2k},p^*)_\infty}{(q^{2k+4}z_2/z_1;q^{2k},p^*)_\infty(q^{-4}z_2/z_1;q^{2k},p^*)_\infty}\\
&\times\ \frac{(q^{2k+4}z_2/z_1;q^{2k},p)_\infty(q^{-4}z_2/z_1;q^{2k},p)_\infty}{(q^{2k+2}z_2/z_1;q^{2k},p)_\infty(q^{-2}z_2/z_1;q^{2k},p)_\infty},\\
\mathcal{A}^2(z_1)\mathcal{A}^{*1}(z_2) &= :\mathcal{A}^2(z_1)\mathcal{A}^{*1}(z_2):\frac{(q^{2k+2}z_2/z_1;q^{2k},p^*)_\infty(q^{-2}z_2/z_1;q^{2k},p^*)_\infty}{(q^{2k+4}z_2/z_1;q^{2k},p^*)_\infty(q^{-4}z_2/z_1;q^{2k},p^*)_\infty}\\
&\times\ \frac{(q^{2k+4}z_2/z_1;q^{2k},p)_\infty(q^{-4}z_2/z_1;q^{2k},p)_\infty}{(q^{2k+2}z_2/z_1;q^{2k},p)_\infty(q^{-2}z_2/z_1;q^{2k},p)_\infty}.
\end{aligned}$$

Here we have used the notation

$$(z;p_1,p_2)_\infty=\prod_{n_1,n_2=0}^{\infty}(1-p_1^{n_1}p_2^{n_2}z).$$

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